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A QR DECOMPOSITION METHOD FOR FLEXIBLE MULTIBODY DYNAMICS

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A QR DECOMPOSITION METHOD FOR FLEXIBLE MULTIBODY DYNAMICS

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ABSTRACT

Deformable components in multibody systems are subject to kinematic constraints the represent mechanical joints and specified motion trajectories. These constraints can, in general, be described using a set of nonlinear algebraic equations that depend on the system generalized coordinates and time. This paper describes an efficient procedure for the computer implementation of the *absolute nodal coordinate formulation* for flexible multibody applications. In the absolute nodal coordinate formulation, no infinitesimal or finite rotations are used as nodal coordinates. The configuration of the finite element is defined using global displacement coordinates and slopes. By using this mixed set of coordinates, beam and plate elements can be treated as isoparametric elements. As a consequence, the dynamic formulation of these widely used elements using the absolute nodal coordinate formulation leads to a constant mass matrix. It is the objective of this study to develop a an efficient computational procedure that exploits this feature. In this procedure, an optimum sparse matrix structure is obtained for the deformable bodies using the QR decomposition. Using the fact that the element mass matrix is constant, a QR decomposition of a modified constant connectivity Jacobian matrix is obtained for the deformable body. A constant velocity transformation is used to obtain an identity generalized inertia matrix associated with the second derivatives of the coordinates used in the absolute nodal coordinate formulation, thereby minimizing the number of non-zero entries of the coefficient matrix that appears in the augmented Lagrangian formulation of the equations of motion of the flexible multibody systems. The computational procedure proposed in this investigation can be used for the treatment of large deformation problems in flexible multibody systems. It has also the advantages of the algorithms based on the floating frame of reference formulations since it allows for easy addition of general nonlinear constraint and force functions.

KEY WORDS: Multibody Dynamics, Finite Element Method, QR Decomposition, Absolute Nodal Coordinate Formulation, Floating Frame of Reference Formulation, Incremental Methods, Large Deformation, Large Rotation.

1. INTRODUCTION

The performance and efficiency of multibody simulation codes depend largely on the selection of the coordinates used to formulate the dynamic equations of multibody systems (Kim and Vanderpoleg, 1986; Mani et al., 1985; and Singh and Likins, 1985). The choice of the coordinates defines the structure of the system equations of motion as well as the numerical procedure required for the solution of these equations. Extensive research efforts have been devoted to examine the effect of the coordinate selection on the complexity of the formulation as well as the efficiency and performance of the computer algorithms used in rigid multibody dynamics. The research in flexible multibody dynamics, on the other hand, has been primarily focused on some fundamental issues related to modeling the dynamic motion of flexible bodies that undergo large displacements (Shabana, 1997; and Wasfy and Noor, 1997). Several finite element formulations have been proposed for the large displacement analysis of flexible multibody systems. Among these formulations are the floating frame of reference method (Shabana, 1989), the incremental methods (Belytschko and Hsieh, 1973; Rankin and Brogan, 1986), and large rotation vector formulations (Simo and Vu-Quoc, 1986). In the floating frame of reference formulation, which is the most widely used method for flexible multibody dynamics, two sets of coordinates are used to define the configuration of the flexible body. The first set is the set of reference coordinates which defines the location and orientation of a selected deformable body coordinate system. The second set is the set of elastic coordinates which defines the deformation of the body with respect to its coordinate system. The floating frame of reference formulation leads to a highly nonlinear mass matrix as the result of the inertia coupling between the rigid body motion and the elastic deformation. This formulation, however, can be used to obtain an exact representation of the rigid body dynamics and leads to zero strains under an arbitrary rigid body

motion, even in the case when non-isoparametric finite elements such as conventional beam and plate elements are used.

Incremental finite element formulations have been successfully used in the large deformation analysis of structural systems. In the incremental methods, the configuration of the finite element is described using the element nodal coordinates. Since non-isoparametric elements can not describe an arbitrary large rotation, the large rotation of the element is represented as sequence of infinitesimal rotations. The infinitesimal rotations can then be described using the conventional element shape function and the element nodal coordinates. It is important to note, however, that when non-isoparametric elements such as beams and plates are used, the incremental methods can not be used to obtain exact representation of the rigid body dynamics, and such methods do not lead to zero strains under an arbitrary rigid body motion. For this reason, the incremental methods are not widely used in flexible multibody computer programs.

The large rotation vector formulations are non-incremental and were developed to circumvent the partial linearization used in the finite element incremental formulations. In the large rotation vector formulations, absolute coordinates and finite rotations are used to define the element configuration. These formulations lead to a simpler inertia matrix and a more complex stiffness matrix. Nonetheless, the mass matrix remains nonlinear when three dimensional elements are used. Large rotation vector formulations also lead to excessive shear forces as the result of the description used for the finite rotation of the element cross section (Shabana, 1997).

As previously pointed, the floating frame of reference formulation is the most widely used method in the dynamic analysis of flexible multibody systems. This formulation has been implemented in several commercial and research computer programs. The popularity of this method is atributed

to the fact that general constraints and forcing functions can be systematically and easily introduced to the formulation and can also be implemented in the computer programs in a straightforward manner. However, the use of the floating frame of reference formulation has been limited to small deformation problems; a limitation that arises as the result of the type of coordinates and the motion description used in this formulation. Recently, a new procedure called the *absolute nodal coordinate formulation* was introduced (Shabana, 1997). In this formulation, a new set of finite element coordinates is employed. This set of coordinates consists of global displacement coordinates and slopes. Using this set of coordinates, beam and plate elements can be treated as isoparametric elements, and as a consequence, exact modeling of the rigid body dynamics can be obtained using the element shape function and the element nodal coordinates. Furthermore, the absolute nodal coordinate formulation leads to zero strains under an arbitrary rigid body motion. When such a formulation is used, some of the fundamental problems encountered when using the floating frame of reference formulation can be avoided. One of these problems is the selection of the deformable body coordinate system (Shabana, 1989). This problem does not arise in the absolute nodal coordinate formulation since the coordinates used in this formulation are all defined in the global system. The absolute nodal coordinate formulation also leads to a constant mass matrix, and as such, an efficient procedure for implementing this formulation in flexible multibody computer programs can be developed.

In the general computational algorithms developed for the nonlinear dynamic analysis of multibody systems, nonlinear constraint equations that describe mechanical joints and specified motion trajectories can be systematically introduced to the dynamic formulation. This provides the flexibility of building computer models for mechanical systems with complex topological structure.

By utilizing sparse matrix algebra, these models can be efficiently simulated and modified. In order to maintain the generality of the dynamic formulation, several of the general purpose computer algorithms developed for the simulation of multibody applications are based on solving the following system of equations (Shabana, 1994):

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{q}}^{\mathrm{T}} \\ \mathbf{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e \\ \mathbf{Q}_d \end{bmatrix} \tag{1}$$

where $\mathbf{M}$ is the mass matrix of the system, $\mathbf{C}_{\mathbf{q}}$ is the Jacobian matrix of the kinematic constraints, $\mathbf{q}$ is the vector of the system generalized coordinates, λ is the vector of Lagrange multipliers, $\mathbf{Q}_e$ is the vector of forces that include external, gravity, Coriolis, centrifugal, and elastic forces, and $\mathbf{Q}_d$ is the vector resulting from the differentiation of the constraint equations twice with respect to time. For instance, the constraint equations can be written in the following form:

$$\mathbf{C}(\mathbf{q}, t) = \mathbf{0} \tag{2}$$

where $\mathbf{C}$ is the vector of constraint functions, and t is time. Upon differentiating the constarint equations twice with respect to time, one obtains

$$\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}} = \mathbf{Q}_d \tag{3}$$

where $\mathbf{Q}_d$ is the vector defined as

$$\mathbf{Q}_d = -\mathbf{C}_{tt} - \left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}} \tag{4}$$

The coefficient matrix of Eq. 1 has a sparse matrix structure. An efficient solution of this system of

equations can be obtained if the number of non-zero entries of the nonlinear coefficient matrix is minimized.

It is the objective of this investigation to develop a new computational procedure for flexible multibody dynamics that exploits the sparse matrix structure presented in Eq. 1. The new procedure is based on the *absolute nodal coordinate formulation* which leads to a constant mass matrix for the deformable body (Shabana, 1997). In the absolute nodal coordinate formulation, global displacement coordinates and slopes are used to define the element configuration. Connectivity conditions between the finite elements used in the discretization of the deformable body can be described using a set of linear algebraic constraint equations. Using the constant Jacobian matrix of these constraint equations and the fact that the system mass matrix is constant, an efficient **QR** decomposition for a constant generalized Jacobian matrix can be obtained only once before the dynamic simulation (Atkinson, 1987; Press et al., 1992; and Strang, 1988). Orthogonal vectors resulting from this decomposition are used to define an identity inertia matrix associated with the coordinates of the deformable body, thereby minimizing the number of non-zero entries of the sparse coefficient matrix that appears in Eq. 1.

In the analysis presented in this investigation we distinguish between two sets of constraints. The first set consists of the constraints that describe the connectivity between the elements of a deformable body in the multibody system. These constraints are assumed to be linear functions of the nodal coordinates of the deformable body. We will refer to these constraints as the *connectivity constraints*. The forces resulting from these connectivity constraints will be systematically eliminated from the equations of motion of the deformable body using the **QR** decomposition of the a generalized constraint Jacobian matrix. The second set of constraints consists of the constraints that

describe the joints between deformable bodies as well as the constraints that describe specified motion trajectories. The forces of these constraints will not be eliminated from the dynamic formulation in order to increase the generality of the algorithm presented in the paper.

This paper is organized as follows. In Section 2, the absolute nodal coordinate formulation is briefly reviewed and the form of the constant inertia matrix associated with the coordinates used in this formulation is defined. In Section 3, the equations of motion of the finite elements expressed in terms of the element connectivity forces are presented. These element equations are used to define the equations of motion of the deformable body. In Section 4, the constraint equations that describe the connectivity of the elements of the deformable body are presented and used to define the connectivity constraint Jacobian matrix. Using this constraint Jacobian matrix, the element connectivity forces are expressed in terms of Lagrange multipliers. Utilizing the fact that the mass matrix of the deformable body is constant, a generalized connectivity Jacobian matrix is defined and its **QR** factors are presented in Section 5. A computational procedure for eliminating the connectivity forces from the equations of motion of the deformable body is presented in Section 6., while the final form of the equations of motion and the solution algorithm of these equations are presented in Section 7. Summary and discussion of the work described in this paper is presented in Section 8.

2. ABSOLUTE NODAL COORDINATE FORMULATION

In the analysis presented in this paper, two dimensional beam elements are used as examples. The procedure developed in this investigation, however, can be also applied to three dimensional beam, plate and shell elements. Figure 1 shows a two-dimensional beam element which has two nodes defined by the points A and B. In the absolute nodal coordinate formulation, global displacements

coordinates and slopes are used as the nodal coordinates. In this formulation, no infinitesimal or finite rotations are used as nodal coordinates. By using the global displacement coordinates and slopes, and a shape function that has a complete set of rigid body modes; beam and plate elements can be treated as isoparametric elements that can be used to obtain exact modeling of the rigid body dynamics (Shabana and Christensen, 1997). In the absolute nodal coordinate formulation, the global position of an arbitrary point on an element j on the deformable body i in the multibody system can be written as

$$\mathbf{r}^{ij} = \mathbf{S}^{ij}\mathbf{e}^{ij} \tag{5}$$

where $\mathbf{r}^{ij}$ is the global position vector of an arbitrary point on the beam element, $\mathbf{S}^{ij}$ is the element shape function which has a complete set of rigid body modes, and $\mathbf{e}^{ij}$ is the vector of absolute nodal coordinates and slopes of the element. Using the preceding equation, the kinetic energy of the finite element can be defined as

$$T^{ij} = \frac{1}{2}\int_{V^{ij}} \rho^{ij}\dot{\mathbf{r}}^{ij\mathrm{T}}\dot{\mathbf{r}}^{ij}\,dV^{ij} \tag{6}$$

where ρ^{ij} and V^{ij} are the mass density and volume of the finite element j of the deformable body i. Using the preceding two equations, it can be shown that the kinetic energy of the finite element can be written as

$$T^{ij} = \frac{1}{2}\dot{\mathbf{e}}^{ij\mathrm{T}}\mathbf{M}^{ij}\dot{\mathbf{e}}^{ij} \tag{7}$$

where $\mathbf{M}^{ij}$ is the constant mass matrix of the element defined as

$$\mathbf{M}^{ij} = \int_{V^{ij}} \rho^{ij} \mathbf{S}^{ij\mathrm{T}} \mathbf{S}^{ij} \, dV^{ij} \tag{8}$$

The fact that the element mass matrix is constant plays a fundamental role in the algorithm developed in this investigation. Note that using the absolute nodal coordinate formulation, the mass matrix remains also constant when three dimensional beams are considered. As a consequence, the vector of Coriolis and centrifugal forces is identically equal to zero. The stiffness matrix, on the other hand, is a highly nonlinear function of the element coordinates, even in the case when small deformation problems are considered.

Since in the absolute nodal coordinate formulation, the coordinates are defined in the global system; there is no reason to justify using different interpolating polynomials for the displacement components. For example, in the case of the beam element shown in Fig. 1, cubic polynomials can be used to describe both components of the displacements defined in Eq. 5. In this case, the element shape function can be defined as

$$\mathbf{S}^{ij} = \begin{bmatrix} 1-3\xi^2+2\xi^3 & 0 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & 0 & l(\xi^3-\xi^2) & 0 \\ 0 & 1-3\xi^2+2\xi^3 & 0 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & 0 & l(\xi^3-\xi^2) \end{bmatrix}^{ij} \tag{9}$$

where l is the length of the element, $\xi = x/l$, and x is the axial coordinate that defines the position of an arbitrary point on the element in the undeformed state. Using this shape function, the vector of nodal coordinates of the element can be defined as

$$\mathbf{e}^{ij} = \left[e_1^{ij} \;\; e_2^{ij} \;\; e_3^{ij} \;\; e_4^{ij} \;\; e_5^{ij} \;\; e_6^{ij} \;\; e_7^{ij} \;\; e_8^{ij} \right]^{\mathrm{T}} \tag{10}$$

where e_1^{ij} and e_2^{ij} are the absolute displacement coordinates of the node at A, e_5^{ij} and e_6^{ij} are the absolute displacement coordinates of the node at B, and

$$\left.\begin{aligned} e_3 &= \frac{\partial r_1(x=0)}{\partial x}, & e_4 &= \frac{\partial r_2(x=0)}{\partial x} \\ e_7 &= \frac{\partial r_1(x=l)}{\partial x}, & e_8 &= \frac{\partial r_2(x=l)}{\partial x} \end{aligned}\right\} \tag{11}$$

and r_1 and r_2 are the components of the vector $\mathbf{r}$. Using the shape function of Eq. 9, the constant mass matrix of the element can be evaluated, using Eq. 8, as

$$\mathbf{M}^{ij} = m^{ij} \begin{bmatrix} \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & -\frac{13l}{420} & 0 \\ 0 & \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & -\frac{13l}{420} \\ \frac{11l}{210} & 0 & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} & 0 \\ 0 & \frac{11l}{210} & 0 & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} \\ \frac{9}{70} & 0 & \frac{13l}{420} & 0 & \frac{13}{35} & 0 & -\frac{11l}{210} & 0 \\ 0 & \frac{9}{70} & 0 & \frac{13l}{420} & 0 & \frac{13}{35} & 0 & -\frac{11l}{210} \\ -\frac{13l}{420} & 0 & -\frac{l^2}{140} & 0 & -\frac{11l}{210} & 0 & \frac{l^2}{105} & 0 \\ 0 & -\frac{13l}{420} & 0 & -\frac{l^2}{140} & 0 & -\frac{11l}{210} & 0 & \frac{l^2}{105} \end{bmatrix} \tag{12}$$

It can be demonstrated that the absolute nodal coordinate formulation leads to exact modeling of rigid body dynamics and does not lead to the linearization of the equations of motion as in the case of

incremental formulations. The equivalence of the absolute nodal coordinate formulation and the floating frame of reference formulation can be demonstrated if local slopes instead of infinitesimal rotations are used as the local elastic coordinates in the floating frame of reference formulation. This equivalence was recently demonstrated by Shabana and Schwertassek (1997).

3. FINITE ELEMENT EQUATIONS OF MOTION

In the dynamic formulation presented in this paper, we consider two types of constraints. The first type of constraints is the constraints that represent the connectivity conditions between the finite elements of a deformable body. These constraints are linear functions of the element nodal coordinates. The second type of constraints is the nonlinear constraints that represent the joints between different deformable bodies, or specified motion trajectories. These nonlinear constraints are described by Eq. 2, and will be introduced to the dynamic formulation using the technique of Lagrange multipliers in order to maintain the generality of the formulation.

The first type of linear constraints that describe the connectivity of the elements of one deformable body will be the subject of discussion in this and the following sections. A non-conventional algorithm that utilizes the fact that the element mass matrix is constant will be presented and its advantages will be discussed. To this end, the equations of motion of the finite element j on the deformable body i are written as

$$\mathbf{M}^{ij}\ddot{\mathbf{e}}^{ij} = \mathbf{F}_e^{ij} + \mathbf{F}_c^{ij} \tag{13}$$

where $\mathbf{F}_e^{ij}$ is the vector of element forces that include the externally applied and gravity forces, and the elastic forces that result from the element deformation, and $\mathbf{F}_c^{ij}$ is the vector of constraint forces

resulting from the connectivity of the element with the other elements used in the finite element discretization of the deformable body. Using the preceding equation, the equations of motion of the deformable body i can be written in terms of the connectivity forces as

$$\mathbf{M}^i \ddot{\mathbf{e}}^i = \mathbf{F}_e^i + \mathbf{F}_c^i \tag{14}$$

where $\mathbf{M}^i$ is the mass matrix of the deformable body given in terms of the mass matrices of its elements as

$$\mathbf{M}^i = \begin{bmatrix} \mathbf{M}^{i1} & & & \\ & \mathbf{M}^{i2} & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & \mathbf{M}^{in_e} \end{bmatrix} \tag{15}$$

in which n_e is the total number of elements of the deformable body. The vectors $\mathbf{e}^i$, $\mathbf{F}_e^i$, and $\mathbf{F}_c^i$ are defined as

$$\mathbf{e}^i = \begin{bmatrix} \mathbf{e}^{i1} \\ \mathbf{e}^{i2} \\ \vdots \\ \mathbf{e}^{in_e} \end{bmatrix}, \quad \mathbf{F}_e^i = \begin{bmatrix} \mathbf{F}_e^{i1} \\ \mathbf{F}_e^{i2} \\ \vdots \\ \mathbf{F}_e^{in_e} \end{bmatrix}, \quad \mathbf{F}_c^i = \begin{bmatrix} \mathbf{F}_c^{i1} \\ \mathbf{F}_c^{i2} \\ \vdots \\ \mathbf{F}_c^{in_e} \end{bmatrix} \tag{16}$$

Note that in these equations, the elements are not assembled and none of the element coordinates is eliminated. The equations of the system, however, are still valid since the connectivity forces appear in these equations.

4. JACOBIAN OF THE CONNECTIVITY CONSTRAINTS

The constraints that describe the connectivity between the finite elements of the deformable body i will be formulated in this investigation using a set of algebraic equations that depend linearly on the nodal coordinates. In this case, these constraint equations can be written in the following form:

$$\mathbf{\Phi}^i(\mathbf{e}^i) = \mathbf{c} \tag{17}$$

where $\mathbf{\Phi}^i$ is the connectivity constraint vector, and $\mathbf{c}$ is a constant vector. Note that the connectivity constraints do not depend on time. For a virtual change of the body coordinates, the preceding equation leads to

$$\mathbf{\Phi}_{\mathbf{e}}^i \delta \mathbf{e}^i = \mathbf{0} \tag{18}$$

where $\mathbf{\Phi}_e^i$ is the Jacobian matrix of the connectivity constraints. The connectivity constraint forces in Eq. 14 can be expressed in terms of the Jacobian matrix of the element connectivity constraints as

$$\mathbf{F}_c^i = -\mathbf{\Phi}_e^{i\mathrm{T}} \boldsymbol{\lambda}_c^i \tag{19}$$

where $\boldsymbol{\lambda}_c^i$ is the vector of Lagrange multipliers associated with the connectivity constraints of the deformable body. Substituting Eq. 19 into Eq. 14, one obtains

$$\mathbf{M}^i \ddot{\mathbf{e}}^i = \mathbf{F}_e^i - \mathbf{\Phi}_e^{i\mathrm{T}} \boldsymbol{\lambda}_c^i \tag{20}$$

Since the mass matrix $\mathbf{M}^i$ is constant and block-diagonal, its inverse can be efficiently calculated onlynce before the dynamic simulation and used to write the preceding equation in the following form:

$$\ddot{\mathbf{e}}^i = \left(\mathbf{M}^i\right)^{-1} \mathbf{F}_e^i - \left(\mathbf{M}^i\right)^{-1} \mathbf{\Phi}_e^{i\mathrm{T}} \boldsymbol{\lambda}_c^i \tag{21}$$

which can be written as

$$\ddot{\mathbf{e}}^i = \left(\mathbf{M}^i\right)^{-1} \mathbf{F}_e^i - \mathbf{A}^i \boldsymbol{\lambda}_c^i \tag{22}$$

where

$$\mathbf{A}^i = \left(\mathbf{M}^i\right)^{-1} \mathbf{\Phi}_e^{i\mathrm{T}} \tag{23}$$

is a constant generalized Jacobian matrix. This matrix, which needs to be evaluated only once before the dynamic simulation, plays a significant role in the computational procedure proposed in this paper.

5. FACTORS OF THE GENERALIZED JACOBIAN

The connectivity constraint forces acting on the elements of the deformable body can be systematically eliminated using the **QR** factors of the generalized Jacobian matrix of Eq. 23. Using thq **QR** decomposition, the generalized Jacobian matrix of Eq. 23 can be written as

$$\mathbf{A}^i = \mathbf{Q}\,\mathbf{R} \tag{24}$$

where **Q** is an orthogonal matrix ($\mathbf{Q}^{\mathrm{T}}\,\mathbf{Q} = \mathbf{I}$), and **R** is an upper-triangular matrix. If $\mathbf{A}^i$ is an $m \times n$ matrix, then **Q** is an $m \times n$ and **R** is $n \times n$ matrix. Since the number of connectivity constraints is

always smaller than the number of coordinates, $m \geq n$. In this case, the preceding equation can be written as (Kim and Vanderploeg, 1986)

$$\mathbf{A}^i = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix} \tag{25}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the partitions of the orthogonal matrix $\mathbf{Q}$. The dimension of $\mathbf{Q}_1$ is $m \times n$, while the dimension of $\mathbf{Q}_2$ is $m \times (m - n)$. Since $\mathbf{Q}$ is an orthogonal matrix, it follows that

$$\mathbf{Q}_1^{\mathrm{T}} \mathbf{Q}_2 = \mathbf{0} \tag{26}$$

It is also clear from Eq. 25 that

$$\mathbf{A}^i = \mathbf{Q}_1 \mathbf{R}_1 \tag{27}$$

Using the preceding three equations, it can be verified that

$$\mathbf{Q}_2^{\mathrm{T}} \mathbf{A}^i = \mathbf{0} \tag{28}$$

This result will be used to eliminate the connectivity forces from the equations of motion of the deformable body as demonstrated in the following section.

6. ELIMINATION OF THE CONNECTIVITY FORCES

Recall that the matrix $\mathbf{A}^i$ is a constant matrix, and therefore, the matrix $\mathbf{Q}_2$ that results from the **QR** decomposition of $\mathbf{A}^i$ is also constant. This fact can be utilized to obtain an efficient procedure for eliminating the connectivity forces and define an identity generalized inertia matrix for the deformable

body. To this end, the following velocity transformation is employed:

$$\dot{\mathbf{e}}^{i} = \mathbf{Q}_2 \dot{\mathbf{q}}^{i} \tag{29}$$

Substituting this coordinate transformation into Eq. 18, and premultiplying by the transpose of the matrix $\mathbf{Q}_2$, one obtains

$$\mathbf{Q}_2^{\mathrm{T}} \mathbf{Q}_2 \ddot{\mathbf{q}}^{i} = \mathbf{Q}_2^{\mathrm{T}} \left(\mathbf{M}^{i}\right)^{-1} \mathbf{F}_e^{i} - \mathbf{Q}_2^{\mathrm{T}} \mathbf{A}^{i} \boldsymbol{\lambda}_c^{i} \tag{30}$$

Since the columns of the matrix $\mathbf{Q}_2$ represent a set of orthogonal vectors, one has

$$\mathbf{Q}_2^{\mathrm{T}} \mathbf{Q}_2 = \mathbf{I} \tag{31}$$

Substituting Eqs. 28 and 31 into Eq. 30, one obtains

$$\ddot{\mathbf{q}}^{i} = \mathbf{Q}_2^{\mathrm{T}} \left(\mathbf{M}^{i}\right)^{-1} \mathbf{F}_e^{i} \tag{32}$$

which shows that the generalized inertia matrix associated with the set of coordinates $\mathbf{q}^i$ is the identity matrix. As a consequence, a number of non-zero entries equal to the number of the coordinates in the set $\mathbf{q}^i$ needs to be stored in order to define the inertia matrix of the deformable body. Note also that in Eq. 32, the forces of the connections between the finite elements of the deformable body are eliminated. Equation 32, therefore, reduces to the simple form

$$\ddot{\mathbf{q}}^{i} = \mathbf{Q}_e^{i} \tag{33}$$

where $\mathbf{Q}_e^{i}$ is the vector of generalized forces defined as

$$\mathbf{Q}_e^i = \mathbf{Q}_2^{\mathrm{T}} \left(\mathbf{M}^i\right)^{-1} \mathbf{F}_e^i \tag{34}$$

Equation 33 is in a form suitable for implementation in general purpose flexible multibody computer programs that exploit sparse matrix techniques.

7. SYSTEM EQUATIONS AND NUMERICAL PROCEDURE

Non-linear constraints between deformable bodies can be formulated in the absolute nodal coordinate formualation using Eq. 2. In this case, the augmented form of the equations of motion takes the following form:

$$\begin{bmatrix} \mathbf{I} & \mathbf{C}_{\mathbf{q}}^{\mathrm{T}} \\ \mathbf{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e \\ \mathbf{Q}_d \end{bmatrix} \tag{35}$$

where in this case the generalized inertia matrix of the system reduces to an identity matrix. In Eq. 35, $\mathbf{C}_{\mathbf{q}}$ is the Jacobian matrix of the kinematic constraints that describe the joints between deformable bodies as well as specified motion trajectories. The vector $\mathbf{q}$ in Eq. 35 is the vector of the system coordinates which can be written as

$$\mathbf{q} = \left[\mathbf{q}^{1\mathrm{T}} \quad \mathbf{q}^{2\mathrm{T}} \quad \ldots. \quad \mathbf{q}^{n_b\mathrm{T}}\right]^{\mathrm{T}}$$

where $\mathbf{q}^i$ is the vector of generalized coordinates of the deformable body i, and n_b is the total number of bodies in the system. Equation 35 can be efficiently solved for the accelerations and Lagrange multipliers using sparse matrix techniques (Duff et al., 1986; Press et al., 1992). The vector of Lagrange multipliers can be used to determine the generalized joint forces. Using the constraint

Jacobian matrix of the joint constraints $\mathbf{C_q}$, a set of independent coordinates can be identified. The accelerations associated with the independent coordinates can be integrated forward in time in order to determine the independent velocities and coordinates. Using the independent coordinates, Eq. 2 can be solved for the dependent coordinates using a Newton-Raphson algorithm. Dependent velocities can be determined using the velocity kinematic relationships (Shabana, 1994)

8. SUMMARY AND DISCUSSION

Several finite element formulations have been proposed for the dynamic analysis and simulation of flexible multibody systems. Among these formulations are the floating frame of reference method, the incremental methods, and the large rotation vector formulations. The floating frame of reference formulation is the most widely used method for the dynamic analysis of flexible multibody systems since such a formulation allows easy addition of general constraint and force functions. The use of this method, however, has been limited to small deformation problems because of the nature of the generalized coordinates used. Incremental methods, on the other hand, had less acceptance in the multibody community because of the linearized kinematic equations used to describe the overall motion of the finite element when these methods are used. Because of this kinematic description, exact modeling of the rigid body dynamics can not be obtained using the incremental methods when non-isoparametric finite elements such as conventional beam and plate elements are used. The coordinates used in the non-incremental large rotation vector formulations include finite rotations which are described using interpolation polynomials in a similar manner to the displacement coordinates. Such a motion description leads to excessive shear forces that lead to serious numerical and fundamental modeling problems.

In this investigation, a new computational finite element procedure is developed for the computer-aided analysis of flexible multibody systems. This procedure, which is based on the absolute nodal coordinate formulation, leads to an optimum sparse matrix structure and allows for easy addition of constraints and forcing functions, thereby maintaining the main advantages of the algorithms based on the floating frame of reference formulation. Furthermore, the new procedure can be used for the large deformation analysis of flexible multibody systems, and as such, it does not suffer from the limitation of the floating frame of reference formulation. In the absolute nodal coordinate formulation, global displacement coordinates and slopes are used to define the element configuration. Infinitesimal or finite rotations are not used as nodal coordinates. By using this set of coordinates, exact modeling of the rigid body dynamics can be obtained using the element shape function and the nodal coordinates. As a consequence of using these coordinates, beam elements can be treated as isoparametric elements and an arbitrary rigid body displacement leads to zero strains. The absolute nodal coordinate formulation also leads to a constant mass matrix, and as a result, the vector of Coriolis and centrifugal forces is identically equal to zero.

In this investigation, an efficient implementation of the absolute nodal coordinate formulations for flexible multibody applications is described. In the procedure described in this paper, advantage is taken of the fact that the element inertia matrix is constant. Two different sets of kinematic constraints are considered. The first set consists of the constraints that describe the connectivity of the elements of a deformable body in the multibody system. These constraints which are linear functions of the nodal coordinates are systematically eliminated using the procedure described in this paper. The second set consists of the constraints that describe mechanical joints between deformable bodies in the system as well as specified motion trajectories. These constraints are, in general,

nonlinear and they are not eliminated from the dynamic formulation in order to increase the generality of the procedure described in this paper. The equations of motion of the finite elements are first formulated in terms of a redundant set of coordinates. These equations are expressed in terms of the connectivity forces which are expressed in terms of the constraint Jacobian matrix and lagrange multipliers. A constant generalized Jacobian matrix is obtained using the inverse of the constant element mass matrix. A **QR** decomposition is used to determine the factors of the generalized Jacobian matrix. Using the orthogonal matrix in the **QR** decomposition, a velocity transformation which consists of orthogonal column vectors is developed and used to express the coordinates of the elements of the deformable body in terms of a reduced set of generalized coordinates. The inertia matrix associated with the reduced set of coordinates is the identity matrix. Using this result, an optimum sparse matrix structure for the equations of motion of the flexible multibody system can be defined. The resulting system of equations of motion can be solved using the numerical procedure described in Section 7. Note also that despite the fact that the element connectivity constraint forces are eliminated from the final form of the equation of motion, these forces can be easily computed using the procedure described in this paper. To this end, the nodal accelerations of the deformable body i can be computed from the generalized accelerations of the body using the time derivative of Eq. 29. These nodal accelerations can be substituted in Eq. 14 in order to determine the connectivity force vector $\mathbf{F}_c^i$.

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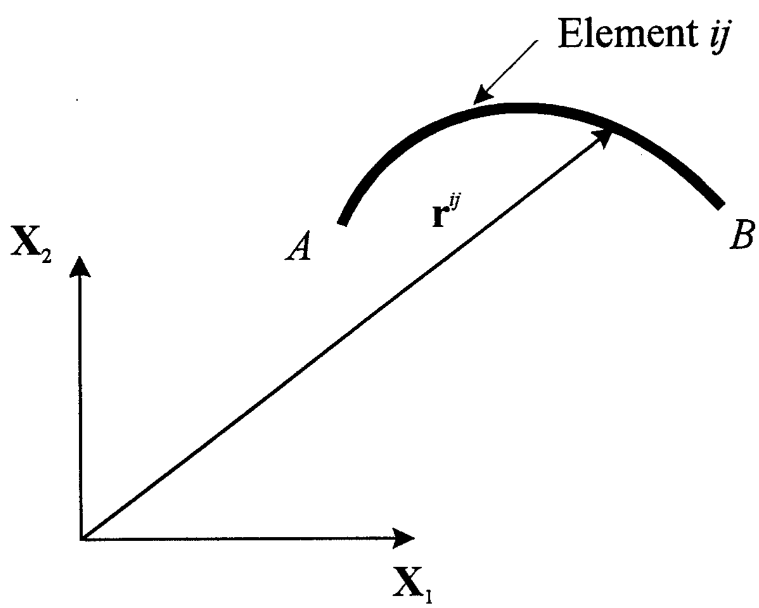

Figure. 1 Absolute Nodal Coordinate Formulation